To Appear in GRG Journal

DIAS-TP-77-36

A RECONSTRUCTION IN MINKOWSKIAN SPACE-TIME OF EINSTEIN'S ASSEMBLY OF TEST PARTICLES

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ABSTRACT

We construct the energy-momentum tensor in Minkowskian space-time for Einstein's collisionless system of test particles moving in concentric circles, and obtain the 4-force necessary to preserve equilibrium. We derive a tensor field, satisfying the linearized Einstein equations, which is consistent with the applied 4-force. If the particles are contained within a sphere then outside the sphere we show that the tensor field is a linearized Schwarzschild field <u>with a cosmological constant</u> (this constant being the "potential energy" calculated on the surface of the sphere).

1. Introduction.

Einstein [1] was the first to consider an assembly of test particles each moving in a circle about a common centre in the gravitational field of all the others. His objective at the time was to construct a realistic source for the Schwarzschild field in which the Schwarzschild event horizon would be naturally enveloped by matter. The stability of the system was investigated by Gilbert [2] and was further discussed by Harrison et al. [3]. The collapse of the system under gravity has been examined by Datta [4] and Bondi [5]. A simple derivation of Einstein's energy-momentum tensor (outlined briefly below) was described by Hogan [6]. Recently a thin shell of such particles has been studied, using Israel's [7] theory of shells, by Evans [8].

It is generally assumed that each particle is maintained in a circular orbit by the gravitational attraction of the combined masses and thus the system remains in equilibrium. This explanation requires some investigation for, as Synge [9] has pointed out, "in relativity there is no such thing as the force of gravity, for gravity is built into the structure of space-time, and exhibits itself in the curvature of space-time". We are interested to know precisely how the circular motion of each particle is maintained in just such a way that the system of particles is in equilibrium. We provide here a partial answer to this question by reconstructing Einstein's assembly of particles in Minkowskian space-time and by deriving a Lorentz covariant tensor field on Minkowskian space-time which provides the necessary stresses and energy to maintain the equilibrium of the assembly and the circular orbits of the test particles. An interesting by-product of this approach is the natural appearance of the cosmological constant. This raises questions concerning a a "linearized" version of Einstein's assembly of particles, which are briefly discussed at the end of the paper.

2. Einstein's Concentric Time-Like Helices.

The entire discussion which follows takes place in Minkowskian space-time M_4. The line-element of M_4 in spherical polar coordinates and time reads [1]

$$ds^2 = -dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2) + dt^2 \equiv g_{ij}\, dx^i dx^j \,, \tag{2.1}$$

with $0 \leq r < +\infty$, $0 \leq \theta \leq \pi$, $0 \leq \phi < 2\pi$, $-\infty < t < +\infty$ and the identification $(x^1, x^2, x^3, x^4) = (r, \theta, \phi, t)$. We may introduce an orthonormal tetrad $\lambda^i_{(a)}$, $a = 1,2,3,4$ as a basis for M_4. At any (finite) event it is given by

$$\lambda^i_{(1)} = (1,0,0,0)\,, \quad \lambda^i_{(2)} = (0, r^{-1}, 0, 0)\,, \quad \lambda^i_{(3)} = (0,0,(r\sin\theta)^{-1},0)\,,$$
$$\lambda^i_{(4)} = (0,0,0,1)\,, \tag{2.2}$$

so that

$$g_{ij}\, \lambda^i_{(a)} \lambda^j_{(b)} = \eta_{ab} = \mathrm{diag.}(-1,-1,-1,+1) \,. \tag{2.3}$$

Consider the time-like world-tube Σ: $r = R$ (constant) generated by the integral curves of $\lambda^i_{(4)}$ (see figure 1). This represents, in M_4, the history of a 2-sphere of radius R at rest with respect to an observer having 4-velocity $\lambda^i_{(4)}$. We shall describe such an observer as being in the macroscopic rest-frame (MRF) of the 2-sphere. Inside Σ consider the non-compact 4-volume to be filled with concentric time-like helices having the time-like geodesic $x^i = (0,0,0,t)$ as common axis. Let the equations of a typical time-like helix be (extending the range of ϕ)

$$r = \text{constant} < R \,, \quad \theta = \pi/2 \,, \quad \phi = \omega t \,, \tag{2.4}$$

where ω is a parameter which may depend upon r. The corresponding time-like unit tangent vector is

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1) We use units for which $c = G = 1$. Latin indices take values 1,2,3,4.

$$V^i = \gamma(0,0,\omega,1) \,, \quad \gamma = (1-\omega^2 r^2)^{-1/2}, \quad \omega r < 1 \,, \tag{2.5}$$

so that $V^i V_i = 1$. Each helix represents the history of a particle travelling in a circle with centre $r = 0$, radius $r < R$, and constant angular velocity $\omega(r)$ relative to the MRF [2]. If m is the (constant) proper-mass of each particle then the 4-momentum of a typical particle is $mV^i \equiv p^i$ with V^i given by (2.5).

Figure 1

The kinematical energy-momentum tensor inside Σ is derived briefly as follows (cf. [6] for details): At any event P inside Σ choose a 4-velocity vector V^i, equation (2.5) being a typical example. Let dS_0 be a normal 3-element to V^i at P. If dS is any other 3-element at P polarised by the unit normal n^i then, by the projection formula,

$$dS_0 = n_i V^i\, dS = n_i p^i m^{-1} dS \,. \tag{2.6}$$

The flux of 4-momentum across dS_0 is

$$F^i = N p^i\, dS_0 \,, \tag{2.7}$$

where N is the proper particle number density (independent of p^i). Combining (2.6) and (2.7) we have

$$F^i = N m^{-1} p^i p^j n_j\, dS = t^{ij} n_j\, dS \,, \tag{2.8}$$

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2) It therefore belongs to Type IIc in Synge's [10] classification of time-like helices in M_4. For each helix (2.4), i.e. for every constant value of r, the first and second curvatures have the constant values $r\gamma^2\omega^2$ and $\gamma^2\omega$ respectively. The third curvature vanishes.

with

$$t^{ij} = N m^{-1} p^i p^j \quad . \qquad (2.9)$$

If this quantity is averaged over all possible directions of p^i at P,with the restriction that p^i should be tangent to a time-like helix at P,we obtain (cf. [6]) the kinematical energy-momentum tensor at P whose non-vanishing components are

$$T^4{}_4 = mN(1-\omega^2 r^2)^{-1} \equiv \rho^* \quad , \qquad (2.10a)$$

$$T^2{}_2 = T^3{}_3 = -\tfrac{1}{2}\rho^* \omega^2 r^2 \quad . \qquad (2.10b)$$

We can rewrite this as

$$T^{ij} = \rho^* \{ \lambda^i_{(4)} \lambda^j_{(4)} + \tfrac{1}{2}\omega^2 r^2 (\lambda^i_{(2)} \lambda^j_{(2)} + \lambda^i_{(3)} \lambda^j_{(3)}) \} \quad , \qquad (2.11)$$

from which we see that

$$T^{ij}\lambda_{(4)j} = \rho^* \lambda^i_{(4)} \quad , \quad T^{ij}\lambda_{(\alpha)j} = -\tfrac{1}{2}\rho^*\omega^2 r^2 \lambda^i_{(\alpha)}, \ \alpha = 2,3 \ , T^{ij}\lambda_{(1)j} = 0. \qquad (2.12)$$

Thus ρ^* is the kinematical mass density measured in the MRF and $S_{(2)} = S_{(3)} = -\frac{1}{2}\rho^*\omega^2 r^2$ are the principal tensions (hoop stresses) in the θ and ϕ directions. The principal stress $S_{(1)}$ in the radial direction vanishes. The principal stresses are smaller in magnitude than the mass density,thus satisfying the "strong energy condition" [11].

We note that if m^* and N^* denote the mass of a particle and the particle number density respectively measured in the MRF then these are related to m and N above by the well-known formulae

$$m^* = \gamma m \quad , \quad N^* = \gamma N \quad , \qquad (2.13)$$

and thus we see that

$$\rho^* = m^* N^* \quad , \qquad (2.14)$$

verifying the interpretation given to ρ^* above. Also we see that

$$T^i{}_i = mN \quad , \qquad (2.15)$$

which is the total microscopic proper-density of proper-mass,in agreement with a general result of Synge [12]. Indeed Synge has proved that for a collection of masses (which may be undergoing collisions),the macroscopic density equals the sum of all the microscopic proper-densities of proper-mass, reduced by the sum of the three principal stresses. This result is verified in our case by substituting (2.11)into (2.15) to obtain

$$\rho^* = mN - S_{(2)} - S_{(3)} \quad . \qquad (2.16)$$

Using (2.12) we derive

$$T^{ij}{}_{|j} = -g^{ij} V^*_{,j} \equiv f^i \quad , \qquad (2.17)$$

where the stroke indicates covariant differentiation with respect to the metric g_{ij} given by (2.1),the comma denotes partial differentiation with respect to x^i and

$$V^* \equiv -\int_0^r \rho^* \omega^2 r \, dr \quad . \qquad (2.18)$$

From (2.17) we conclude that the system described by T^{ij} does not conserve 4-momentum (cf.[12],p.284) since an "applied" 4-force f^i is present. We have chosen $V^* = 0$ when $r = 0$ for when $r = 0$ the helices degenerate into a geodesic which represents the history of a particle on which no 4-force vector is defined. The energy-momentum tensor

$$\mathcal{T}^{ij} \equiv T^{ij} + V^* g^{ij} \quad , \qquad (2.19)$$

satisfies the conservation equation

$$\mathcal{T}^{ij}{}_{|j} = 0 \quad , \qquad (2.20)$$

and thus the combined system of the particles,plus the scalar field (2.18) which provides the necessary 4-force to maintain equilibrium, conserves 4-momentum. From the eigenvector equations for (2.19) we find that the total energy density in the MRF is

$$\mathcal{D}^* = \rho^* + V^* \tag{2.21}$$

which suggests we call V^* the potential energy of the system in the MRF. There is, naturally, no kinetic energy term here. The principal stresses are now

$$t_{(1)} = V^* \quad , \quad t_{(2)} = t_{(3)} = s_{(2)} + V^* \quad . \tag{2.22}$$

We now have a radial stress present in the form of a tension since in a physically reasonable situation $V^* < 0$, since ρ^* would decrease to zero at $r = R$. Inside Σ

$$\mathcal{T}^{ij} = T^{ij} + V^*_{(I)} g^{ij} \equiv \mathcal{T}^{ij}_{(I)} \quad , \tag{2.23}$$

with $V^*_{(I)}$ given by (2.18) with $r \leq R$, and outside Σ

$$\mathcal{T}^{ij} = V^*_{(E)} g^{ij} \equiv \mathcal{T}^{ij}_{(E)} \quad , \tag{2.24}$$

with

$$V^*_{(E)} = -\int_0^R \rho^* \omega^2 r \, dr = \text{constant} \quad , \tag{2.25}$$

since $\rho^* = 0$ for $r > R$. Both (2.23) and (2.24) satisfy the conservation law (2.20).

We shall demonstrate that $\mathcal{T}^{ij}$ is connected to the presence of a Lorentz covariant tensor field γ_{ij} on M_4.

3. The "Applied" Tensor Field.

We shall consider a symmetric tensor field γ_{ij} on M_4 which satisfies the Lorentz covariant linearized Einstein field equations, so that if γ_{ij} were considered a small quantity (which is not necessarily the case in this paper) then

$$g_{ij} + \gamma_{ij} \quad , \tag{3.1}$$

would be an approximate solution to Einstein's generally covariant field equations. On account of spherical symmetry we can simplify the work by taking all components of γ_{ij}, except γ_{11} and γ_{44}, to be zero. We then construct the Lorentz covariant (linearized) Einstein tensor[3] which in the present coordinate system has the non-vanishing components

$$G^1{}_1 = r^{-2}\left(r\gamma_{44,1} + \gamma_{11}\right) \quad , \tag{3.2a}$$

$$G^2{}_2 = G^3{}_3 = \tfrac{1}{2}\left(\gamma_{44,11} + r^{-1}\gamma_{44,1} + r^{-1}\gamma_{11,1}\right) \quad , \tag{3.2b}$$

$$G^4{}_4 = r^{-2}\left(r\gamma_{11,1} + \gamma_{11}\right) \quad . \tag{3.2c}$$

We easily check that the Lorentz covariant, twice-contracted Bianchi identities,

$$G^i{}_{j|i} \equiv 0 \quad , \tag{3.3}$$

are satisfied by (3.2). The stroke here indicates, as always, covariant differentiation with respect to the metric given by (2.1). Thus, on account of (2.20) we postulate, as field equations for γ_{ij},

$$G^i{}_j = -8\pi \mathcal{T}^i{}_j \quad , \tag{3.4}$$

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[3] These components are obtained by first calculating the Einstein tensor for the line-element

$$ds^2 = -e^{\alpha} dr^2 - r^2(d\theta^2 + \sin^2\theta \, d\phi^2) + e^{\gamma} dt^2 \quad ,$$

(cf. [9], p.272) and then taking $e^{\alpha} = 1 - \gamma_{11}$, $e^{\gamma} = 1 + \gamma_{44}$ and retaining only first order terms in γ_{11} and γ_{44}.

with J^{ij} given by $J^{ij}_{(I)}$ inside Σ and $J^{ij}_{(E)}$ outside Σ . A straightforward integration of (3.2a) and (3.2c) yields,in general,

$$\gamma_{11} = r^{-1}\int_0^r r^2 G^4{}_4 \, dr \quad , \tag{3.5a}$$

$$\gamma_{44} = \gamma_{11} + \int_0^r r\left(G^1{}_1 - G^4{}_4\right) dr \quad , \tag{3.5b}$$

and substituting these into (3.2b) we obtain a "consistency condition",

$$G^2{}_2 = G^3{}_3 = (2r)^{-1}\frac{\partial}{\partial r}\left(r^2 G^1{}_1\right) \quad . \tag{3.6}$$

In terms of $J^i{}_j$ this reads

$$J^2{}_2 = J^3{}_3 = (2r)^{-1}\frac{\partial}{\partial r}\left(r^2 J^1{}_1\right) \quad , \tag{3.7}$$

which,using (2.19),becomes

$$T^2{}_2 + V^* = T^3{}_3 + V^* = (2r)^{-1}\frac{\partial}{\partial r}\left(r^2 V^*\right) \quad . \tag{3.8}$$

Expanding the derivative here and using (2.18) we find that this equation is identically satisfied.

To obtain $\gamma^{(I)}_{ij}$,i.e. γ_{ij} inside Σ ,we replace $G^i{}_j$ by $-8\pi J_{(I)}{}^i{}_j$ in (3.5) and perform the indicated integrations. This requires one to make a choice of $\rho^*(r)$,which up to now is only constrained to be positive inside Σ and zero outside Σ ,and of $\omega(r)$. A simple (but unphysical)choice is ρ^*= const., ω = const.. The integrations in (3.5) are easily carried out to yield

$$\gamma^{(I)}_{11} = -2\frac{M}{R}\left(\frac{r}{R}\right)^2\left(1 - \frac{3}{10}\omega^2 r^2\right) \quad , \tag{3.9a}$$

$$\gamma^{(I)}_{44} = \gamma^{(I)}_{11} + 3\frac{M}{R}\left(\frac{r}{R}\right)^2 \quad , \tag{3.9b}$$

where $M = \frac{4\pi}{3}\rho^* R^3$.

To obtain $\gamma^{(E)}_{ij}$,i.e. γ_{ij} outside Σ ,we replace $G^i{}_j$ with $-8\pi J_{(E)}{}^i{}_j$ in (3.5). This can be carried out without a specification of ρ^* . The field equations (3.4) now read

$$G^i{}_j = -8\pi V^*_{(E)}\,\delta^i_j \quad , \tag{3.10}$$

which are the Lorentz covariant linearized Einstein equations with a cosmological constant $\Lambda = 8\pi V^*_{(E)}$,which in general is negative (see remark following equation (2.22)). Upon integration we obtain

$$\gamma^{(E)}_{11}(r) = \left\{R\gamma^{(I)}_{11}(R) + \tfrac{1}{3}\Lambda R^3\right\}r^{-1} - \tfrac{1}{3}\Lambda r^2 = -Ar^{-1} - \tfrac{1}{3}\Lambda r^2 \, , \tag{3.11a}$$

$$\gamma^{(E)}_{44}(r) = \gamma^{(E)}_{11}(r) - \gamma^{(I)}_{11}(R) + \gamma^{(I)}_{44}(R) = -Ar^{-1} - \tfrac{1}{3}\Lambda r^2 + B \, . \tag{3.11b}$$

We see immediately that $\gamma^{(E)}_{ij}$ joins continuously to $\gamma^{(I)}_{ij}$ on Σ . In the case of small γ_{ij} ,substitution of (3.11) into (3.1) shows that from our viewpoint the approximate solution obtained in this way,of the generally covariant Einstein field equations,outside Σ is the linearized Schwarzschild solution with a cosmological constant. The constant A is twice the "Schwarzschild mass" of the source. For the case ρ^*= const., ω=const. (cf.(3.9)) it is given by $2\left(M + \frac{1}{5}MR^2\omega^2\right)$. It is interesting to note that the second term inside the brackets is precisely the same as the kinetic energy of a 2-sphere of radius R and mass M rotating with angular velocity ω .It is the contribution to the Schwarzschild mass of the scalar field V^*. The constant B ,which is small of order γ_{ij} ,can be transformed away by the infinitesmal translation $\bar{r} = r$, $\bar{\theta} = \theta$, $\bar{\phi} = \phi$, $\bar{t} = \left(1 + \frac{B}{2}\right)t$.

The Lorentz covariant linearized Riemann tensor for γ_{ij} is obtained in the same manner as the corresponding Einstein tensor (3.2). The non-vanishing components are:

$$R_{2323} = r^2 \sin^2\theta\, \gamma_{11} \quad , \quad R_{1414} = -2^{-1}\gamma_{44,11} \quad ,$$

$$R_{1212} = 2^{-1} r \gamma_{11,1} \quad , \quad R_{2424} = -2^{-1} r \gamma_{44,1} \quad , \qquad (3.12)$$

$$R_{3131} = \sin^2\theta\, R_{1212} \quad , \quad R_{3434} = \sin^2\theta\, R_{2424} \quad .$$

Evaluating these for $\gamma_{ij}^{(E)}$ and projecting them onto the orthonormal tetrad (2.2) we find that

$$R^{(E)}_{(2323)} = -\frac{1}{3}\Lambda - Ar^{-3} \quad , \quad R^{(E)}_{(1414)} = \frac{1}{3}\Lambda + Ar^{-3} \quad ,$$

$$R^{(E)}_{(1212)} = -\frac{1}{3}\Lambda + \frac{1}{2}Ar^{-3} \quad , \quad R^{(E)}_{(2424)} = \frac{1}{3}\Lambda - \frac{1}{2}Ar^{-3} \quad , \qquad (3.13)$$

$$R^{(E)}_{(3131)} = R^{(E)}_{(1212)} \quad , \quad R^{(E)}_{(3434)} = R^{(E)}_{(2424)} \quad .$$

These results may be summarised as

$$R^{(E)}_{(abcd)} = \frac{1}{3}\Lambda\left(\eta_{ad}\eta_{bc} - \eta_{ac}\eta_{bd}\right) + O(r^{-3}) \quad . \qquad (3.14)$$

Hence, for large values of r, in the frame (2.2), the field R_{ijkl} outside Σ becomes a De Sitter field (cf. [9], p.256) with constant curvature $K = \frac{1}{3}\Lambda < 0$. We note that in the special case $\rho^* =$ const., $\omega =$ const. we find $\omega^2 \propto -K$, giving the microscopic parameter ω a macroscopic interpretation.

4. Discussion.

In Einstein's original model he considered the histories of his test particles to be circular geodesics of the line-element (in our notation)

$$ds^2 = -e^{\alpha}dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2) + e^{\gamma}dt^2 \quad , \qquad (4.1)$$

where α and γ are functions of r. In our reconstruction the test

particles have world-lines which are time-like helices confined to the world-tube Σ in M_4. They are not therefore geodesic world-lines in M_4 (nor are they geodesics of the metric (3.1)). It is in this regard that our approach differs fundamentally from Einstein's. We require the scalar field V^* to support the circular motion of the particles and the equilibrium of the system, whereas the gravitational field was sufficient to support Einstein's model. As a consequence of the introduction of V^* we find the cosmological constant occuring in the region exterior to the source (where Einstein obtained the Schwarzschild field without the cosmological term).

Throughout our work we assume ρ^* and ω to be functions of r with r = const. on each helix (and thus ω = const. on each helix). In section 3 we chose to illustrate the special case ρ^* = const., ω = const., in which we obtained the interesting result that the Schwarzschild mass of the source is the mass of a 2-sphere of radius R and density ρ^* plus the kinetic energy of such a sphere rotating with angular velocity ω.

For small γ_{ij} the viewpoint expressed in this paper is one way of regarding the linear approximation (3.1). We see that it differs considerably from the exact generally covariant solution to the problem given in [1].

I wish to thank Professor J. T. Lewis and Dr. D. H. Tchrakian for useful discussions.

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Legend for figure:

The world-tube Σ with a typical time-like helix C contained therein. The central time-like geodesic C' ($r = 0$) is also shown.

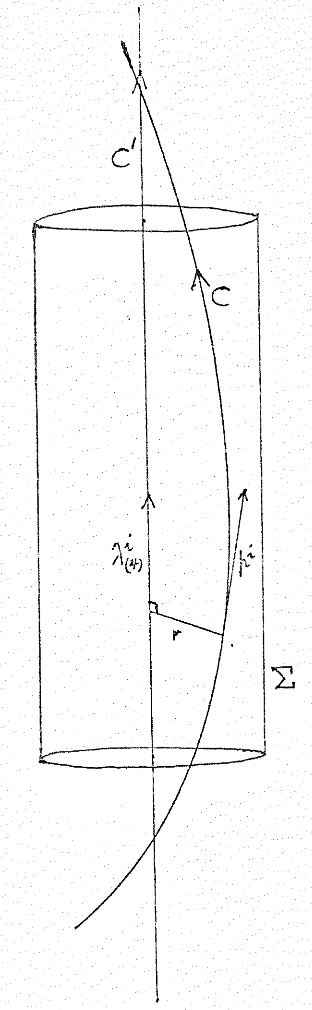

Figure 1